Let $f : I \to \mathbb{R}$ be C^n, and $a \in I$. We define the **n-th order Taylor polynomial of f at a** as the polynomial

$$p^f_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

If the context is clear, we may write $p_{n,a}(x)$ instead of $p^f_{n,a}(x)$. We also defined the "remainder" $r^f_{n,a}(x)$ (or $r_{n,a}(x)$) of the Taylor polynomial, which is the difference between f and its n-th order Taylor polynomial approximation at a:

$$r^f_{n,a}(x) = f(x) - p^f_{n,a}(x).$$

We have proven a few facts about polynoimal approximations in class:

(i) $p_{n,a}(x)$ is a *good n-th order approximation of f at a*. That is,

$$\lim_{x\to a} \frac{r_{n,a}(x)}{(x-a)^n} \left(= \lim_{x\to a} \frac{f(x) - p_{n,a}(x)}{(x-a)^n} \right) = 0.$$

In fact, it is the *only* n-th order polynomial that is a good n-th order approximation of f at a.

(ii) $(r_{n,a})^{(k)}(a) = 0$ for $k = 0, 1, \ldots, n$. As $r_{n,a}(x) = f(x) - p_{n,a}(x)$, this can also be written

$$(p_{n,a})^{(k)}(a) = f^{(k)}(a).$$

(iii) If f is also C^{n+1}, then for $x > a$, there exists some $c \in (a, x)$ such that

$$r_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

For $x < a$, there exists some $c \in (x, a)$ such that the above equation holds.

Problem 1

Suppose p is a good n-th order approximation of f at a:

$$\lim_{x\to a} \frac{f(x) - p(x)}{(x-a)^n} = 0.$$

Show that p is a good k-th order approximation of f at a for all $k = 0, 1, \ldots, n-1$ as well.

Try to do the following problem with as little aid from calculators as possible. You may find the following calculations useful:

$$2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32, 2^6 = 64, 2^7 = 128, 2^8 = 256, 2^9 = 512, 2^{10} = 1024.$$

$$3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, 3^6 = 729, 3^7 = 2187, 3^8 = 6561.$$

$$2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5040, 8! = 40320.$$

Problem 2

1. Find the n-th order Taylor polynomial approximation of cos at $a = 0$.
2. Using fact (iii), find a large enough n so that the nth-order Taylor polynomial of cos at $a = 0$ approximates $\cos(1)$ with an error of less than 10^{-3}. That is, find an n so that

$$|r_{n,a}(1)| \le 10^{-3}.$$

3. Calculate $\cos(1)$ correct to 3 decimal places.

Let $x \in \mathbb{R}$. We defined the **open ball of radius** r **around** x, $B_r(x)$, as the set $(x-r, x+r)$. Given a set $U \subseteq \mathbb{R}$, and a point $a \in \mathbb{R}$, we say:

- a is an **interior point** of U if there exists $r > 0$ so that $B_r(a) \subseteq U$.
- a is a **boundary point** of U if for every $r > 0$, we have $B_r(b) \cap U \neq \emptyset$ and $B_r(b) \cap U^c \neq \emptyset$.

The set of interior points of U is denoted U^{int}, and the set of boundary points of U is denoted ∂U.

Problem 3
Find examples of sets $U \subseteq \mathbb{R}$ which:

1. Have no interior points, but have boundary points.
2. Have no boundary points.
3. Have countably infinitely many boundary points.
4. Have uncountably infinitely many boundary points, and countably infinitely many interior points.

Problem 4
Can a set $U \subseteq \mathbb{R}$ have finitely many interior points?

We say a set $U \subseteq \mathbb{R}$ is:

- **Open** if $U^{\text{int}} = U$.
- **Closed** if $\partial U \subseteq U$.

Problem 5
Find examples of sets $U \subseteq \mathbb{R}$ which:

1. Are open and closed.
2. Are open but not closed.
3. Are closed but not open.
4. Are neither open nor closed.

Problem 6
Show that $U \subseteq \mathbb{R}$ is open if and only if U^c is closed.

Recall we have proven the following in class:

- If $\{U_i\}_{i \in I}$ is an arbitrary collection of open sets, then $\bigcup_{i \in I} U_i$ is also open.
- If $U_1, U_2, \ldots, U_n$ is a *finite* collection of open sets, then $\bigcap_{i=1}^{n} U_i$ is also open.

Problem 7
Find an infinite collection of open sets whose intersection is not open.

Problem 8

1. If $\{C_i\}_{i\in I}$ is an arbitrary collection of closed sets, show that $\displaystyle\bigcap_{i\in I} C_i$ is also closed.
2. If $C_1, C_2, \ldots, C_n$ is a *finite* collection of closed sets, show that $\displaystyle\bigcup_{i=1}^{n} C_i$ is also closed.
3. Show that finiteness is necessary in 2. In other words, find an infinite collection of closed sets whose union is not closed.